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Vacuum leaks in extra dimensions

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Abstract

There is a semiclassical instability in models of compactified spaces with Casimir or Monopole compactifications. Previous claims of stability are wrong because they have been based upon metric ansätze that are singular ($g_{00} \rightarrow 0$) in the limit of large size of the internal space. A non-singular metric ansatz is proposed that demonstrates explicitly this instability. With the new ansatz g_{00} is free of singularities in the limit of large internal dimensions, and the kinetic term for the 4-dimensional gravitational field is independent of the scale of the extra dimensions.

1 Introduction

If the fundamental theory of nature is a "higher-dimensional" one with extra spatial dimensions, it is necessary to hide the extra dimensions. The usual mechanism for hiding the extra dimensions is to assume that they form a compact internal space that has a physical size small enough to escape detection. For currently available energies at accelerators, this requires a size smaller than the Fermi length of about 10^{-16}cm. This would not be surprising, since in almost all extra-dimensional theories the fundamental length scale is set by the Planck length, $l_{Pl} \equiv G_N^{1/2} = 1.616 \times 10^{-33}$cm. In the limit that the physical size of the internal space is smaller than the physical size of the external space, it is possible to dimensionally reduce the system (integrate over the extra dimensions) and obtain an "effective" $3+1$-dimensional theory.

Theories with extra spatial dimensions are many and varied. However all have common features of relevance for cosmology:

There are large spatial dimensions and small spatial dimensions. The assumption that the extra dimensions form a compact space is quite reasonable since if the Universe is closed ($\Omega > 1$), the three observed spatial dimensions form a compact space (a 3-sphere, S^3). The remarkable thing is that there is such a disparity in the sizes – 10^{-33}cm for the internal space and more than 10^{28}cm for the external space.

The fundamental constants we observe today are not truly fundamental. In theories with extra dimensions the truly fundamental constants are the ones in the higher-dimensional theory. The constants that appear in the effective four dimensional theory are the result of integration over the extra dimensions. If the volume of the extra dimensions would change, so would the "observed" constants.

The internal dimensions are static. Since any change in the size of the internal space would result in changes in the observed constants, the extra dimensions must be static, or have changed very little since the time of primordial nucleosynthesis.

The curious cosmology that emerges is one that has some dimensions large and expanding, and some dimensions small and static. Since expansion (or contraction) is the generic behavior expected, the challenge for cosmologists involves constructing models that have static extra dimensions. The basic approach is to assume that the higher dimensional theory is that of gravity plus a cosmological constant.[1] The extra dimensions are held static due to the interplay between the cosmological constant and either classical[2] or quantum[3] fields. Although the true mechanism in more complicated theories such as superstring models might be more complex, there must be some vacuum stress keeping the extra dimensions static. The toy models studied here may very well be relevant.

In the models that have been studied, the present ground state size of the extra dimensions is stable against small fluctuations of the size of the internal space, but not necessarily stable against large dilatations of the compact space.[4] This leads to the possibility that there is a semiclassical instability that would start as a quantum fluctuation of the internal space from its ground state value to a size larger than some critical value, followed by an evolution of the internal space (and the external space) in an expansion that is approximately exponential in time. Thus the present vacuum of a static internal space is a "false vacuum" and the "true vacuum" is exponential expansion of all dimensions. There is some probability that the radius of the internal space could "leak" through a barrier and begin to grow.

This semiclassical instability was first studied by Frieman and Kolb,[4] who also calculated the lifetime of the vacuum against nucleation of a bubble of true vacuum. The calculation was performed by expressing the radius of the internal space as a scalar field, and using the well-known methods of the decay of the false vacuum in quantum field theory. This calculation was later criticized by Maeda,[5] who claimed that quantum gravitational effects not considered in the Frieman-Kolb calculation could be important. The quantum gravitational effects arise because the effective 4-dimensional gravitational action depends upon the size of the internal space. This dependence may be removed by a different ansatz for the metric. Maeda argues that this ansatz leads to a vacuum that is semiclassically stable.

In this paper we show that the metric ansatz employed by Maeda is inappropriate to study the problem of semiclassical stability because of a singularity in the metric. We propose a different ansatz for the metric that is free of the uncertainties of quantum gravity present in the Frieman-Kolb ansatz and free of the singularity problems in the Maeda ansatz. With this new improved ansatz the vacuum is semiclassically unstable.

In the next section we review two compactification schemes, "Casimir" compactification due to the interplay between a cosmological constant and the quantum Casimir effect, and monopole compactification due to the interplay between a cosmological constant and the monopole configuration of a Maxwell field. In the third section the equations of motion are developed for three ansätze for the metric. Different choices for the normalization of the scalar field proportional to the size of the internal space lead to different conclusions about stability. The correct physical choices lead to instability. The paper ends with a concluding section.

2 General Approach

We will start with a theory of gravity in $N = D + 4$ dimensions with a cosmological constant Λ (and in the monopole compactification scheme a Maxwell field). The gravitational part of the action is

$$S = -\frac{1}{16\pi\bar{G}} \int d^N x \sqrt{-g}[R + 2\Lambda], \tag{1}$$

where $\bar{G}$ is the gravitational constant in $D + 4$ dimensions, related to Newton's constant G by $\bar{G} = GV_D^0$ with V_D^0 the present volume of the internal space.

Since the basic goal is to study cosmological solutions to the Einstein equations, it is convenient to express the equations in the form[6]

$$R_{MN} = 8\pi\bar{G}S_{MN}, \tag{2}$$

where S_{MN} can be expressed in terms of a stress-tensor T_{MN} and a cosmological constant Λ

$$S_{MN} = T_{MN} - \frac{1}{D+2} g_{MN} T^P{}_P - \frac{1}{D+2} \frac{\Lambda}{8\pi\bar{G}} g_{MN}. \tag{3}$$

We will assume that the metric is block diagonal, i.e, we can perform a $1+3+D$-dimensional split. In this case the non-vanishing components of the stress tensor are proportional to the metric tensor with components given by

$$\begin{aligned} T_{00} &\equiv \rho g_{00} \\ T_{ij} &\equiv -p_3 g_{ij} \\ T_{\mu\nu} &\equiv -p_D g_{\mu\nu}. \end{aligned} \tag{4}$$

In terms of $\rho,\ p_3$, and p_D,

$$\begin{aligned}
S_{00} &= \frac{8\pi\bar{G}}{D+2}\left[(D+1)\rho + 3p_3 + Dp_D - \rho_\Lambda\right] g_{00} \\
S_{ij} &= \frac{8\pi\bar{G}}{D+2}\left[\rho + (D-1)p_3 - Dp_D + \rho_\Lambda\right] g_{ij} \\
S_{\mu\nu} &= -\frac{8\pi\bar{G}}{D+2}\left[\rho - 3p_3 + 2p_D + \rho_\Lambda\right] g_{\mu\nu},
\end{aligned} \tag{5}$$

where $\rho_\Lambda = \Lambda/8\pi\bar{G}$.

Different compactification schemes result in different forms for $\rho,\ p_3,\ p_D$. In the Casimir compactification scheme the cosmological constant is balanced by one-loop corrections to the action due to vacuum fluctuations of matter fields. For simplicity assume that the internal space is a D-sphere of radius b and volume $V_D = \Omega_D b^D$. In the limit that the radius of the internal space is much smaller that the radius of the 3-space, vacuum fluctuations give contributions[3]

$$\begin{aligned}
\rho &= c_1/\Omega_D b^{4+D} \\
p_3 &= -c_1/\Omega_D b^{4+D} \\
p_D &= 4c_1/D\Omega_D b^{4+D} \qquad\qquad \text{(Casimir)}.
\end{aligned} \tag{6}$$

Here c_1 is a constant that depends on the number of fields present in the theory. For a single massless scalar field on S^7, $c_1 = 8.16 \times 10^{-4}$.

The monopole compactification scheme involves the introduction of an antisymmetric tensor field of rank $D-1$ with field strength $F_{M,N,\ldots,Q}$ of rank D. The stress tensor for F is given by

$$T_{MN} = F_{MP\ldots Q}F_N{}^{P\ldots Q} - \frac{1}{2D}g_{MN}F_{SP\ldots Q}F^{SP\ldots Q}. \tag{7}$$

This field has a natural Freund-Rubin ansatz[7] on the D-sphere:

$$F_{MN\ldots Q} = \begin{cases} f\sqrt{g^{(D)}}\varepsilon_{\mu\nu\ldots\varsigma} & \text{on the internal space} \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

where f is b-dependent, and $g^{(D)}$ is the determinant of the metric for a unit D-sphere. The Bianchi identities require $f = f_0/b^D$, where f_0 is a constant (independent of

b). With the Freund-Rubin monopole ansatz in the limit $a \gg b$

$$\begin{aligned} \rho &= f_0^2/2b^{2D} \\ p_3 &= -f_0^2/2b^{2D} \\ p_D &= f_0^2/2b^{2D} \qquad \text{(Monopole).} \end{aligned} \tag{9}$$

With all metric ansätze the assumption is made that the curvature of the external space is negligible, and that the metric for the internal space is that of a D-sphere. In general, the metric will be written in the form

$$ds^2 = g_{mn}dx^m dx^n - b^2(t)\tilde{g}^{(D)}_{\mu\nu}dx^\mu dx^\nu \tag{10}$$

where $b(t)$ is the radius of the D-sphere, and $\tilde{g}^{(D)}_{\mu\nu}$ is the metric for the D-sphere of unit radius.

The simplest ansatz for g_{mn} is to assume that it is the usual 4-dimensional Robertson-Walker metric with $g_{00} = 1$, and $g_{ij} = -a^2(t)\tilde{g}^{(3)}_{ij}$ where $a(t)$ is the radius of the 3-sphere. This is the ansatz used by Frieman and Kolb. With this ansatz the non-zero components of the Ricci tensor are (dot denotes d/dt)

$$\begin{aligned} -R_{00} &= 3\frac{\ddot{a}}{a} + D\frac{\ddot{b}}{b} \\ -R_{ij} &= \left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + D\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{2}{a^2}\right] g_{ij} \\ -R_{\mu\nu} &= \left[\frac{\ddot{b}}{b} + (D-1)\frac{\dot{b}^2}{b^2} + 3\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{D-1}{b^2}\right] g_{\mu\nu}. \end{aligned} \tag{11}$$

Since the curvature of the external space will be neglected relative to the curvature of the internal space, the $2/a^2$ term in the ij equation will be ignored hereafter.

Using the three equations (00, ij, $\mu\nu$) $R_{MN} = 8\pi\tilde{G}S_{MN}$ it is possible to search for static solutions.[8] The search for static solutions involves finding the value(s) of b that result in all time derivatives equal to zero. For the Casimir case the Einstein equations give

$$\begin{aligned} 3\frac{\ddot{a}}{a} + D\frac{\ddot{b}}{b} &= -\frac{8\pi\tilde{G}}{D+2}\left[\frac{(D+2)c_1}{\Omega_D}b^{-4-D} - \rho_\Lambda\right] \\ \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + D\frac{\dot{a}}{a}\frac{\dot{b}}{b} &= -\frac{8\pi\tilde{G}}{D+2}\left[\frac{(D+2)c_1}{\Omega_D}b^{-4-D} - \rho_\Lambda\right] \end{aligned}$$

$$\frac{\ddot{b}}{b} + (D-1)\frac{\dot{b}^2}{b^2} + 3\frac{\dot{a}}{a}\frac{\dot{b}}{b} = \frac{8\pi\bar{G}}{D+2}\left[\frac{4(D+2)c_1}{D\Omega_D}b^{-4-D} + \rho_\Lambda\right] - \frac{D-1}{b^2}. \tag{12}$$

Note that the curvature term for S^D ($\propto 1/b^2$) has been moved to the right hand side of the $\mu\nu$ equation where it belongs, as it acts as a stress. The static solution obtains when the right hand side of Eqns. (12) vanish. This occurs at $b = b_0$, where

$$b_0^2 = \frac{8\pi c_1(4+D)}{D(D-1)}l_{Pl}^2 = \left(\frac{\Omega_D}{(D+2)c_1}\rho_\Lambda\right)^{-2/(4+D)}. \tag{13}$$

In terms of b_0, Eq.(12) can be written as

$$\begin{aligned}
3\frac{\ddot{a}}{a} + D\frac{\ddot{b}}{b} &= -(D-1)b_0^{-2}\left[\frac{D}{4+D}\left(\frac{b_0}{b}\right)^{4+D} - \frac{D}{4+D}\right] \\
\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + D\frac{\dot{a}}{a}\frac{\dot{b}}{b} &= -(D-1)b_0^{-2}\left[\frac{D}{4+D}\left(\frac{b_0}{b}\right)^{4+D} - \frac{D}{4+D}\right] \\
\frac{\ddot{b}}{b} + (D-1)\frac{\dot{b}^2}{b^2} + 3\frac{\dot{a}}{a}\frac{\dot{b}}{b} &= (D-1)b_0^{-2}\left[\frac{4}{4+D}\left(\frac{b_0}{b}\right)^{4+D} + \frac{D}{4+D}\right. \\
&\qquad \left. - \left(\frac{b_0}{b}\right)^2\right].
\end{aligned} \tag{14}$$

Of course at $b = b_0$ the right hand side of all the equations vanish and $a = b$ =constant is a solution.

There are two other interesting solutions to the system of equations. At $b = 2^{1/(D+2)}b_0$ the r.h.s. of the $\mu\nu$ equation vanishes, while the r.h.s. of the 00 and ij equation are equal to

$$H_3^2 = \frac{D(D-1)}{4+D}b_0^{-2}\left[1 - 2^{-(D+4)/(D+2)}\right]. \tag{15}$$

This value of b results in a solution with b static and $a(t) \propto \exp(H_3 t/\sqrt{3})$. There is also another interesting solution to the equations in the limit $b \gg b_0$. If $b \gg b_0$, the right hand sides of Eqs.(14) are all equal to

$$H_N^2 = b_0^{-2}\frac{D(D-1)}{4+D}, \tag{16}$$

which results in the solution $a(t) \propto b(t) \propto \exp(H_N t/\sqrt{3})$.

It is most easy to visualize the behavior of the scale factors by constructing a "potential" for b. The $\mu\nu$ equation can be cast in the form of the classical equation of motion of a scalar field under the influence of some potential. For instance, if we choose to define a scalar field $\chi = \ln(b/b_0)$, then the $\mu\nu$ equation becomes

$$\ddot{\chi} + (D-1)\dot{\chi}^2 + 3\frac{\dot{a}}{a}\dot{\chi} = (D-1)b_0^{-2}\left[\frac{4}{4+D}e^{-(4+D)\chi} + \frac{D}{4+D} - e^{-2\chi}\right]. \quad (17)$$

With the identification of the r.h.s. of Eq.(17) as $-dV(\chi)/d\chi$, the equation appears as the classical equation for the evolution of a scalar field χ in an expanding universe with a friction term. The potential is given by

$$V(\chi) = (D-1)b_0^{-2}\left[\frac{4}{(4+D)^2}e^{-(4+D)\chi} - \frac{D}{4+D}\chi - \frac{1}{2}e^{-2\chi} + \frac{D^2+8D+8}{2(4+D)^2}\right] \quad (18)$$

where the constant term has been added to make $V(0) = 0$. This potential is shown in Fig. 1. The three interesting solutions are easy to identify. The static minimum at $b = b_0$ ($\chi = 0$) is stable against small perturbations. The local maximum of the potential corresponds to the solution with b static and a expanding exponentially. It is an unstable solution. Finally, at large b the potential is unbounded from below, and the solution of the equations correspond to exponential expansion of both a and b.

The monopole compactification scheme[2] leads to a potential with all the features of Casimir compactification.[9]

The potential suggests that if b was ever larger than some critical value, it (and a) would expand forever. This suggests that there is a semiclassical instability in the compactification models. Imagine that today $b = b_0$. There should be quantum fluctuations of b about b_0. If the fluctuation is large enough, b will not relax to b_0, but would be "over the hump" and expand forever. To actually study the possibility of semiclassical tunnelling of the vacuum, it is necessary to correctly normalize the scalar field. This is the subject of the next section.

3 Scalar field dynamics

Everyone agrees that the key to the choice of the normalization of the scalar field is to have the kinetic term of the scalar in the "correct" form, but there is disagreement on what the correct form is. Here we present the two forms that have been used involving different ansätze for the metric, and mention problems with each choice. Then we propose a third ansatz free of the problems of the previous ones. This ansatz leads to a vacuum that is semiclassically unstable. In the final section we calculate the rate for vacuum leaks.

The first choice we will consider is the one made by Frieman and Kolb. The metric is of the form

$$ds^2 = dt^2 - a^2(t)d\vec{l}_3^2 - b^2(t)d\vec{l}_D^2, \tag{19}$$

where $d\vec{l}_\psi^2$ is the proper distance for the ψ-sphere and a (b) is the radius of the external (internal) space. The non-vanishing components of the Ricci tensor are given by Eq.(11). The Ricci scalar is

$$-R = 6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} + 2D\frac{\ddot{b}}{b} + D(D-1)\frac{\dot{b}^2}{b^2} + 6D\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{D(D-1)}{b^2}, \tag{20}$$

and

$$\sqrt{g} = a^3(t)b^D(t)\sqrt{\tilde{g}^{(D)}}\sqrt{\tilde{g}^{(3)}}. \tag{21}$$

Upon integration over the extra dimensions, the effective 4-dimensional action is

$$\begin{aligned} S_4 &= -\frac{1}{16\pi G}\int d^4x\sqrt{\tilde{g}^{(3)}}a^3(t)\left(\frac{b}{b_0}\right)^D\left[R_4 - 2D\frac{\ddot{b}}{b}\right. \\ &\qquad\qquad \left. -D(D-1)\frac{\dot{b}^2}{b^2} + 6D\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \ldots\right], \end{aligned} \tag{22}$$

where $\ldots$ includes the cosmological constant, the curvature term for S^D, and the other terms from Casimir or monopole effects. R_4 is the 4-dimensional Ricci scalar (the first two terms of R). Note that the 4-dimensional Newton's constant, $G = \tilde{G}/V_D^0$ enters. Due to the presence of the $(b/b_0)^D$ term, the first term is 4-dimensional Einstein gravity *only* at $b = b_0$. If $b \neq b_0$ the "effective" Newton constant would change. If b is not static, the dynamics of theory is fundamentally different from 4-dimensional Einstein gravity. This is not a fault of the ansatz, but it is a fundamental feature of the higher-dimensional theory.

Upon integration by parts, the 4-dimensional action contains a term

$$S_k = -D(D-1)\frac{m_{Pl}^2}{16\pi}\int d^4x\sqrt{\tilde{g}^{(3)}}a^3(t)\left(\frac{b}{b_0}\right)^{D-2}\left(\frac{\dot{b}}{b_0}\right)^2. \tag{23}$$

If the scalar field ϕ is defined as

$$\phi \equiv \left[\frac{D-1}{2\pi D}\right]^{1/2}\left(\frac{b}{b_0}\right)^{D/2} m_{Pl}, \tag{24}$$

it will have a canonical (but for a sign) kinetic term. This is the choice for the normalization proposed by Frieman and Kolb.

With this choice for ϕ, the $\ddot{b}$ equation becomes

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{\dot{\phi}^2}{\phi} = -\frac{dV(\phi)}{d\phi}, \tag{25}$$

which results in a potential given by

$$\begin{aligned} V(\phi) \;=\; & \left(\frac{D(D-1)}{8\pi(D+4)}\right)^2 \frac{(D-1)}{c_1} m_{Pl}^4 \left\{ \left(\frac{\phi}{\phi_0}\right)^{-8/D} - \left(\frac{\phi}{\phi_0}\right)^2 \right. \\ & \left. + \frac{D+4}{D-2}\left[\left(\frac{\phi}{\phi_0}\right)^{2(D-2)/D} - 1\right]\right\}, \end{aligned} \tag{26}$$

where $\phi_0 \equiv \phi(b_0)$, and a constant has been added to the potential to make $V(\phi_0) = 0$. This potential is unbounded from below, and the present vacuum $\phi = \phi_0$ is semiclassically unstable.

Although the kinetic term for ϕ is "correct", the kinetic term for gravity is not, due to the $(b/b_0)^D$ term. Maeda claims that the approximation that 4-dimensional quantum-gravitational effects can be ignored is a bad one, and that inclusion of the effects stabilizes the potential.

Maeda removes the offending term by using a different metric ansatz. He performs a conformal rescaling of the 4-dimensional part of the metric by a factor of b^{-D}.[10] The metric used by Maeda is

$$ds^2 = \left(\frac{b(\eta)}{b_0}\right)^{-D}\left[d\eta^2 - a^2(\eta)d\vec{l}_3^{\,2}\right] - b^2(\eta)d\vec{l}_D^{\,2}. \tag{27}$$

This metric results in a Ricci tensor with non-vanishing components given by (prime denotes $d/d\eta$)

$$
\begin{aligned}
-R_{00} &= \left(\frac{b}{b_0}\right)^D \left[3\frac{a''}{a} - \frac{D}{2}\frac{b''}{b} - \frac{3D}{2}\frac{b'}{b}\frac{a'}{a} + \frac{D(D+3)}{2}\frac{b'^2}{b^2}\right] g_{00} \\
-R_{ij} &= \left(\frac{b}{b_0}\right)^D \left[\frac{a''}{a} - \frac{D}{2}\frac{b''}{b} - \frac{3D}{2}\frac{a'}{a}\frac{b'}{b} + 2\frac{a'^2}{a^2} + \frac{D}{2}\frac{b'^2}{b^2}\right] g_{ij} \\
-R_{\mu\nu} &= \left(\frac{b}{b_0}\right)^D \left[\frac{b''}{b} - \frac{b'^2}{b^2} + 3\frac{a'}{a}\frac{b'}{b} + \left(\frac{b}{b_0}\right)^{-D}\frac{D-1}{b^2}\right] g_{\mu\nu}.
\end{aligned}
\tag{28}
$$

S_{MN} is unchanged from the previous ansatz.[11]. The right hand side of the equations of motion (the three equations involving a', b', a'', and b'') is a factor of $(b_0/b)^D$ times the right hand side of Eq.(14). This extra factor comes from moving $(b/b_0)^D$ to the right hand side of $R_{MN} = 8\pi\bar{G}S_{MN}$. In the limit of large b the right hand side of the equations of motion vanish, so the system admits the solutions a =constant b =constant in the large b limit. This is counter to one's intuition. Starting with a N-dimensional theory of gravity with a cosmological constant, for large scale factors the energy density should be dominated by vacuum energy,[12] and static solutions should not exist. This is the first hint that something is amiss. The trouble is that the time η is not the physical time.

The equation of motion for b'' can be used to find a potential for b similar to the potential of Fig. 1 for the previous ansatz. $dV(b)/db$ for the new ansatz is equal to a factor of $(b/b_0)^{-D}$ times $dV(b)/db$ for the previous ansatz. Again with the definition $\chi = \ln(b/b_0)$ the potential in this case is

$$
\begin{aligned}
V(\chi) &= (D-1)b_0^{-2}\left[\frac{2}{(D+4)(D+2)}e^{-2(D+2)\chi} + \frac{1}{D+4}e^{-D\chi}\right. \\
&\qquad \left. - \frac{1}{D+2}e^{-(D+2)\chi}\right].
\end{aligned}
\tag{29}
$$

The potential is shown in Fig. 2.

The potential in Fig. 2 has a local minimum that is stable against small perturbations, but not against large perturbations. The potential also has the feature that both $dV(\chi)/d\chi$ and $V(\chi)$ approach zero as $\chi \to \infty$. This gives the impression that there is a static (approximately static) solution for b infinite (large). This result is due to the fact that as $b \to \infty$, $g_{00} \propto b^{-D} \to 0$. If we define a time t that has $g_{00} = 1$

$$(b/b_0)^{-D}d\eta^2 = dt^2, \tag{30}$$

then $a' = (b/b_0)^{-D}\dot{a}$ and $b' = (b/b_0)^{-D}\dot{b}$. It is possible to have a and b expanding exponentially in "time t" but static in "time η." The basic point is that one must specify the "time" by which something is said to be static. With the choice of η as time, the time slicing becomes singular in the large b limit. The physical time is t, not η.

With the new ansatz for the metric, the Ricci scalar is given by

$$-R = b^{-D}\left[6\frac{a''}{a} - D\frac{b''}{b} + 6\frac{a'^2}{a^2} + \frac{D(D+4)}{2}\frac{b'^2}{b^2} - 3D\frac{a'}{a}\frac{b'}{b} + b^D\frac{D(D-1)}{b^2}\right], \tag{31}$$

and

$$\sqrt{g} = b^{-2D}(t)a^3(t)b^D(t)\sqrt{\tilde{g}^{(D)}}\sqrt{\tilde{g}^{(3)}}. \tag{32}$$

The factor of b^{-D} in $\sqrt{g}$ cancels the factor of b^D from R, and the explicit factor of b will not be present. Upon integration over the extra dimensions, the effective four-dimensional action becomes

$$\begin{aligned} S_4 &= -\frac{1}{16\pi G}\int d^4x\sqrt{\tilde{g}^{(3)}}a^3(t)\left[\bar{R}_4 + D\frac{b''}{b}\right. \\ &\qquad \left. -\frac{D(D+4)}{2}\frac{b'^2}{b^2} + 3D\frac{a'}{a}\frac{b'}{b} + \ldots\right], \end{aligned} \tag{33}$$

where $\bar{R}_4$ is the Ricci scalar calculated from the metric $\bar{g}_{mn}$: $ds^2 = \bar{g}_{mn}dx^m dx^n = d\eta^2 - a^2(\eta)d\vec{l}^2$. The first term in Eq.(33) looks like the canonical Einstein-Hilbert action. Of course, just because an expression has the familiar letters does not mean that it has the same physics of gravity. With the new ansatz, the true metric for the 3+1-dimensional space is *not* $\bar{g}_{mn}$, but rather $(b/b_0)^{-D}\bar{g}_{mn}$.

Upon integration by parts, the action contains a term

$$S_k = -\frac{D(D+2)}{2}\frac{m_{Pl}^2}{16\pi}\int d^4x\sqrt{\bar{g}}\left(\frac{b'^2}{b^2}\right). \tag{34}$$

A scalar field

$$\sigma = \sigma_0\ln(b/b_0) = \left[\frac{D(D+2)}{16\pi}\right]^{1/2} m_{Pl}\ln(b/b_0) \tag{35}$$

will have a canonical action for a minimally-coupled scalar field. With this definition for σ, the b'' equation becomes

$$\sigma'' + 3\frac{\dot{a}}{a}\sigma' - \frac{\sigma'^2}{\sigma_0} = -\frac{dV(\sigma)}{d\sigma}, \tag{36}$$

with $V(\sigma)$ given by

$$\begin{aligned} V(\sigma) \;&=\; (D-1)\sigma_0^2 b_0^{-2}\left[\frac{2}{(D+4)(D+2)}e^{-2(D+2)\sigma/\sigma_0}\right. \\ &\qquad \left. + \frac{1}{D+4}e^{-D\sigma/\sigma_0} - \frac{1}{D+2}e^{-(D+2)\sigma/\sigma_0}\right]. \end{aligned} \tag{37}$$

The potential $V(\sigma)$ is similar to the potential of Fig. 2. $V(0)$ and $V(\infty)$ are degenerate global minima of the potential. Note that the potential for $V(\sigma)$ appears to be stable against tunnelling, since the potential energy at the minimum ($\sigma = 0$) is equal to the energy at $\sigma = \infty$. The nucleation of a bubble with $\sigma \geq 0$ would, in the course of its evolution toward large b, violate energy for an infinite amount of "time." In other words the Euclidean action, S_E, is infinite. This would imply that the bubble nucleation rate per four volume, $d\Gamma/dV_4 \propto \exp(-S_E)$ is zero. However the four volume V_4 is the four volume of the metric $\bar{g}_{mn}$, which does not describe the physical four volume, e.g., η is not the physical time. Since $d\eta = (b/b_0)^{D/2}dt$,

$$\frac{d\Gamma}{dt} = \left(\frac{b}{b_0}\right)^{D/2}\frac{d\Gamma}{d\eta}, \tag{38}$$

and in the large b limit the decay rate per *physical* four volume may be finite even if S_E is infinite.

Another way to view the problem is that the microphysics of quantum mechanics is determined by the clock time t, *not* η. Since $dt = (b/b_0)^{-D/2}d\eta$, the quantum uncertainty $\Delta E\Delta t = \Delta E\Delta\eta(b/b_0)^{-D/2}$ implies that in the large b limit, it *is* possible to violate energy for an infinite amount of time η. The time slicing is singular at large b, and since time and energy are conjugate variables, one must be cautious about using energy conservation arguments at large b.

We conclude that the metric ansatz used by Maeda is inappropriate to study the semiclassical stability question.

It is possible to find a metric ansatz that has canonical kinetic terms for both the gravitational field and the scalar field, and also has $g_{00} = 1$. We propose the ansatz

$$ds^2 = dt^2 - b^{-2D/3}(t)a^2(t)d\vec{l}_3^2 - b^2(t)d\vec{l}_D^2. \tag{39}$$

With this ansatz the non-vanishing components of the Ricci tensor are

$$\begin{aligned}
-R_{00} &= \left[3\frac{\ddot{a}}{a} - 2D\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{D(D+3)}{3}\frac{\dot{b}^2}{b^2}\right]g_{00} \\
-R_{ij} &= \left[\frac{\ddot{a}}{a} - \frac{D}{3}\frac{\ddot{b}}{b} + 2\frac{\dot{a}^2}{a^2} + \frac{D}{3}\frac{\dot{b}^2}{b^2} - D\frac{\dot{a}}{a}\frac{\dot{b}}{b}\right]g_{ij} \\
-R_{\mu\nu} &= \left[\frac{\ddot{b}}{b} - \frac{\dot{b}^2}{b^2} + 3\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{D-1}{b^2}\right]g_{\mu\nu}
\end{aligned} \tag{40}$$

the Ricci scalar is

$$-R = 6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} + \frac{D(D+3)}{3}\frac{\dot{b}^2}{b^2} - 2D\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{D(D-1)}{b^2} \tag{41}$$

and

$$\sqrt{g} = a^3(t)\sqrt{\tilde{g}^{(D)}}\sqrt{\tilde{g}^{(3)}}. \tag{42}$$

Upon integration over the extra dimensions, the effective 4-dimensional action is

$$\begin{aligned}
S_4 &= -\frac{1}{16\pi G}\int d^4x\sqrt{\tilde{g}^{(3)}}a^3(t)\left[\hat{R}_4 + \frac{D(D+3)}{3}\frac{\dot{b}^2}{b^2}\right. \\
&\qquad\left. -2D\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \ldots\right],
\end{aligned} \tag{43}$$

where $\hat{R}_4$ is the Ricci scalar calculated with the metric $\hat{g}_{mn}$: $ds^2 = \hat{g}_{mn}dx^m dx^n = dt^2 - a^2(t)d\vec{l}_3^2$. Again, the kinetic term for gravity does not depend upon b. Just as with the previous ansatz, $\hat{g}_{ij}$ is not the physical metric, since the true metric for the three space is $(b/b_0)^{-2D/3}\hat{g}_{ij}$.

The kinetic term for b appears in S_4 as

$$S_k = -\frac{D(D+3)}{3}\frac{m_{Pl}^2}{16\pi}\int d^4x\sqrt{\hat{g}}\left(\frac{\dot{b}^2}{b^2}\right). \tag{44}$$

A scalar field

$$\xi = \xi_0 \ln(b/b_0) = \left[\frac{2D(D+3)}{48\pi}\right]^{1/2} m_{Pl}\ln(b/b_0) \tag{45}$$

will have the canonical kinetic term.

With this definition for ξ the scalar potential now reads

$$V(\xi) = (D-1)\xi_0^2 b_0^{-2}\left[\frac{4}{(4+D)^2}e^{-(4+D)\xi/\xi_0} - \frac{D}{4+D}\frac{\xi}{\xi_0} - \frac{1}{2}e^{-2\xi/\xi_0} + \frac{D^2+8D+8}{2(4+D)^2}\right]. \tag{46}$$

This potential has the same functional form as Eq.(18) shown in Fig. 1. The present vacuum ($\xi = 0$) leaks.

4 Conclusions

A metric ansatz that results in a canonical kinetic term for both gravity and the scalar field *and* is regular in the $b \to \infty$ limit has a vacuum where the static ground state is metastable.

It is straightforward to calculate the action for a nucleation of a bubble with $b \neq 0$. It is convenient to approximate the potential of Eq.(46) by a polynomial in the region most relevant for tunnelling (from $\xi = 0$ to $\xi = \xi_t$, where ξ_t is the second zero of $V(\xi)$). For $D = 7$ the potential of Eq.(46) can be fit by

$$V(\xi) \simeq 0.183\Lambda\xi^2 - 0.168\Lambda\xi^3/m_{Pl}. \tag{47}$$

This potential is shown in Fig. 3 as the broken line and compared with the exact form of the potential Eq.(46) shown as the solid line. The coefficients of ξ^2 and ξ^3 were chosen to have the same values of $V(\xi)$ at the maximum and to have the same values of $\xi_t \simeq 0.31$.

Eq.(47) has the general form $V(\xi) = M^2\xi^2/2 - \delta\xi^3/3$. The tunnel action for such a potential has been calculated by Linde,[13] $S_E \simeq 205M^2/\delta^2$, or in terms of Λ and m_{Pl}, $S_E \simeq 294m_{Pl}^2/\Lambda$.

On dimensional grounds the pre-factor must be of order m_{Pl}^4 and the decay rate per unit four volume is

$$\frac{d\Gamma}{dV_4} \simeq m_{Pl}^4 \exp(-S_E) \simeq m_{Pl}^4 \exp(-294m_{Pl}^2/\Lambda). \tag{48}$$

In a matter-dominated Universe the probability for decay becomes of order unity in a time τ given by $\tau^{-4} \simeq d\Gamma/dV_4$, or $\tau \simeq t_{Pl}\exp(74m_{Pl}^2/\Lambda)$. This is longer than the age of the Universe only if $\Lambda \leq 0.53m_{Pl}^2$. Using the relation between Λ and c_1 ($\Lambda \simeq 5.22m_{Pl}^2/c_1$ for $D = 7$), in order for the internal dimensions to stay small for a long enough time requires $c_1 \geq 9.9$. For S^7 a single scalar field contributes $c_1 = 8.16 \times 10^{-4}$, so there must be more than 12,069 scalar fields.

With the ansatz used by Frieman and Kolb, the scalar field potential is given by Eq.(26) and the tunnel action is $S_E = 165m_{Pl}^2/\Lambda$. With the new ansatz the potential is more stable, but still semiclassically unstable.

Finally, it should be emphasized that the four metric $\hat{g}_{mn}$ is not the complete picture. For instance, from the higher-dimensional metric Eq.(39), the *physical* scale factor of the external three-sphere is $b^{-D/3}a$, *not* a which is the scale factor of $\hat{g}_{mn}$. In the limit $b \rightarrow \infty$ the equations of motion with the ansatz Eq.(39) give $a(t) \propto \exp(H_N\sqrt{3+D}t/3)$, and $b(t) \propto \exp(H_N t/\sqrt{3+D})$ where H_N is given by Eq.(16). Although a and b increase at a different rate, the physical radius of the external space, $b^{-D/3}a$ increases as $\exp(H_N t/\sqrt{3+D})$. As expected, the *physical* scale factors increase at the same rate as in the original ansatz $ds^2 = dt^2 - a^2(t)d\vec{l}_3^2 - b^2(t)d\vec{l}_D^2$.

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Figure Captions

Fig. 1: The potential of Eq.(18) for $D = 7$.

Fig. 2: The potential of Eq.(29) for $D = 7$.

Fig 3.: A comparison between the potential of Eq.(46) given by the solid line, and the numerical fit of Eq.(47) given by the broken line. For both cases, $D = 7$.

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Vacuum leaks in extra dimensions

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Abstract

There is a semiclassical instability in models of compactified spaces with Casimir or Monopole compactifications. Previous claims of stability are wrong because they have been based upon metric ansätze that are singular ($g_{00} \to 0$) in the limit of large size of the internal space. A non-singular metric ansatz is proposed that demonstrates explicitly this instability. With the new ansatz g_{00} is free of singularities in the limit of large internal dimensions, and the kinetic term for the 4-dimensional gravitational field is independent of the scale of the extra dimensions.

In a matter-dominated Universe the probability for decay becomes of order unity in a time τ given by $\tau^{-4} \simeq d\Gamma/dV_4$, or $\tau \simeq t_{Pl}\exp(74m_{Pl}^2/\Lambda)$. This is longer than the age of the Universe only if $\Lambda \leq 0.53m_{Pl}^2$. Using the relation between Λ and c_1 ($\Lambda \simeq 5.22m_{Pl}^2/c_1$ for $D=7$), in order for the internal dimensions to stay small for a long enough time requires $c_1 \geq 9.9$. For S^7 a single scalar field contributes $c_1 = 8.16 \times 10^{-4}$, so there must be more than 12,069 scalar fields.

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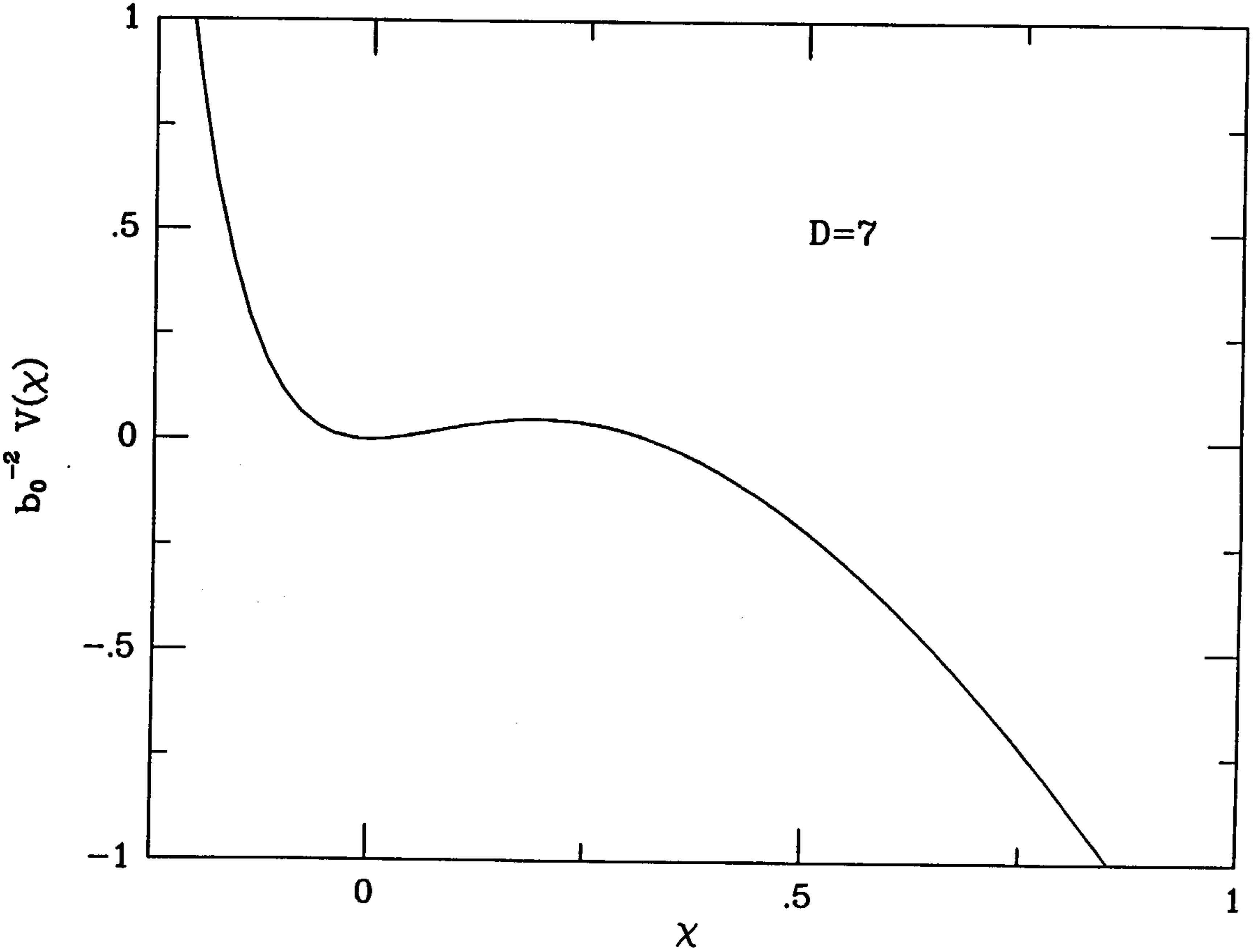

Figure 1

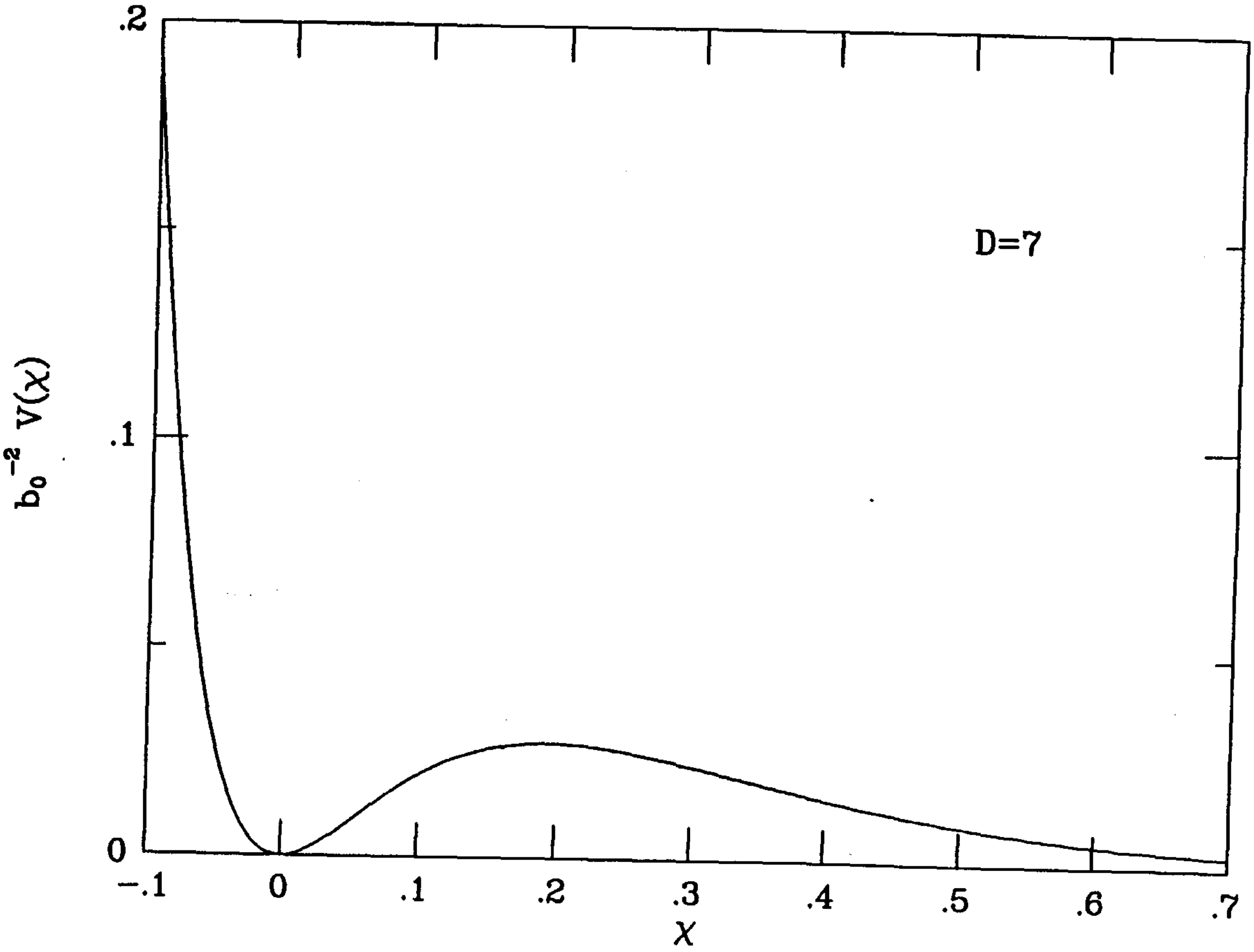

Figure 2

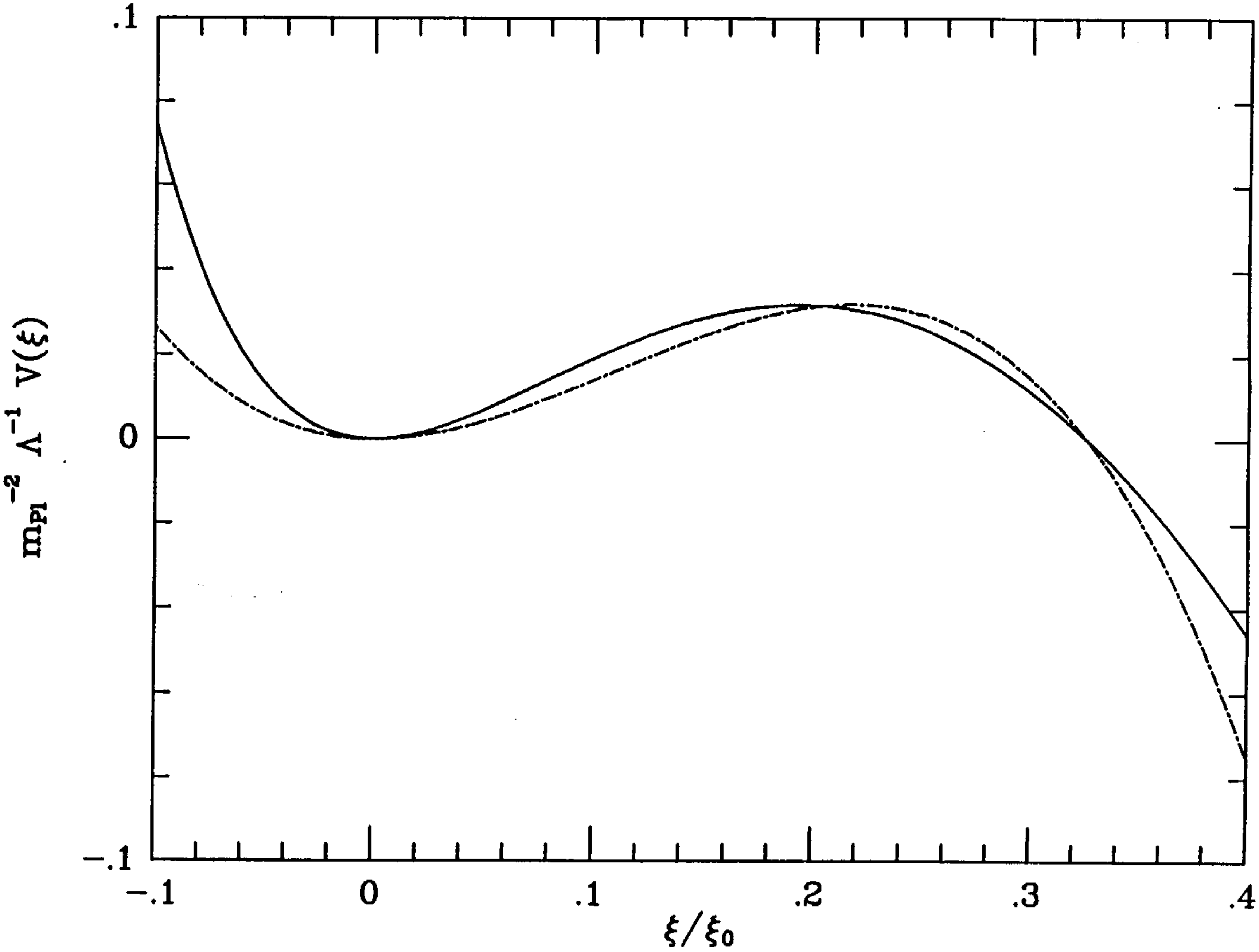

Figure 3